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A MARKOVIAN APPROACH TO ORDERED PEAK STATISTICS

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SUMMARY

A theory based on Markovian principles and transition probability description is presented here to predict the statistics of the ordered peaks in a random process. It takes into account the statistical dependence that exists between the peaks in a single time history. The theory is more general than the other existing theories and, in special cases, it is shown to lead to the independent order statistics as well as to a first passage problem. Digital simulation has been carried out to validate the analytical results. The effects of governing parameters on the statistics of various orders of peaks have also been studied.

KEY WORDS: seismic response; peak factors; order statistics; peak dependence; transition probability; Markov theory

1. INTRODUCTION

Several researchers in the past have attempted to study the distribution of maxima and the statistics of the crossings of a particular level by a random function. A few of them have also studied the statistics of the higher-order peaks which are of particular interest to assess the damage in the mildly non-linear systems due to the isolated and infrequent inelastic excursions (see Reference 1 for details). Gupta and Trifunac[2] proposed the theoretical distributions for these peaks by using the order statistics approach. However, their formulation ignored the statistical dependence that might exist between the unordered peaks occurring in a single realization of a process.

For a random function of finite duration, the probability of the largest maxima occurring below a prescribed barrier is equivalent to the probability of no crossing of the given level by the random function in the specified time. Hence, for investigating the probabilistic estimate of the largest maxima, the study of the probability distribution of the time of the first crossing of a specified level by a random function, popularly known as the first passage problem, has been a widely accepted alternative. Crandall,[3,4] Mark,[5] Cook[6] and Ditlevesen[7] have shown theoretically and through simulation studies that an exponential distribution holds for the first passage problem. Cramer[8] has shown that the assumption of level crossings occurring independently according to a Poisson process is asymptotically exact when the threshold level increases to infinity. Lin[9] and Yang and Shinozuka[10] also obtained reasonable approximations for the first passage problem. Darling and Siegart[11] modelled the problem as a continuous time Markov chain. However, none of the above solutions had a simple form. Vanmarcke[12,13] derived the distribution of first passage time for a normal, stationary random process. Though it has been popular and used extensively, it fails to provide information about the distribution of the higher-order peaks.

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In a companion paper, Basu *et al.*[1] have proposed the distributions of the ordered peaks in a random process by considering the peak dependence through simulated joint density function. In this paper, with a view to provide an analytically sound basis, a Markovian theory based on the transition probability description has been presented to obtain these distributions. This theory is shown, in special cases, to lead to the order statistics formulation based on the independence of the peaks and to take the form of a first passage problem. The theory proposed is strictly valid for a stationary case. However, synthetic accelerograms have been considered in a digital experimentation to see how reasonable would the use of proposed peak factors be in such cases. Both the theoretical and experimental results are seen to be in close agreement. A parametric study has also been carried out to study the effects of the spectral bandwidth and the total number of peaks on the distributions of the ordered peak amplitudes. A detailed comparison of the results of the approach proposed with those based on the simulated joint density and on the independence of peaks has been given elsewhere.[1]

2. TRANSITION PROBABILITY OF PEAKS IN A RANDOM PROCESS

Let us consider a random process $X(t)$ with finite duration T and with n number of peaks or maxima of different orders. Now, two random variables can be defined based on the process $X(t)$. The first one, $N(x)$, is the number of peaks below the value x of $X(t)$, while the second one, $N^*(x)$, is the number of peaks above this value. Thus, both $N(x)$ and $N^*(x)$ are continuous parameter, discrete state random processes. Corresponding to these two random variables, two cases of the transition probability are considered in this paper. For the first case, the transition probability is defined as the probability of occurrence of a particular number of peaks below a certain amplitude level on the condition that the number of peaks below another level of lower amplitude is known. This will, in turn, lead to the probability that a particular order of peak will be below a given magnitude when it is known that a higher-order peak is already below a certain level of magnitude. For the second case, the transition probability is defined as the probability of occurrence of a particular number of peaks above a certain amplitude level on the condition that the number of peaks above another level of lower amplitude is known. These transition probabilities can be used to derive information about the number of crossings of a particular level or the number of inelastic excursions of the response process in a structure.

To illustrate the first case, let us consider the process $N(x)$, where $N(x) = p,\ p \in I$ implies that p number of peaks are less than or equal to the level x in the time duration T of the process. If the values of this random variable at levels u and $v (u \leqslant v)$ are given by $N(u) = i$ and $N(v) = j \geqslant i$, respectively, then the transition probability from amplitude level u to the level v is denoted by

$$p_{ij}(v, u) = \text{Prob}\{N(v) = j \mid N(u) = i\} \tag{1}$$

In the above expression, i and j are called the state variables, while u and v are termed as space variables. For the second case, $N^*(x) = p,\ p \in I$ implies that p number of peaks are greater than or equal to the level x in the time duration T of the process. Then the transition probability from amplitude level u to the level v is denoted by

$$p^*_{ij}(v, u) = \text{Prob}\{N^*(v) = j \mid N^*(u) = i\}, \quad v \geqslant u,\ i \geqslant j \tag{2}$$

It may be noted that the transition probability is a function of both initial and final levels and not just of the difference between the two. This can be represented mathematically as

$$p_{ij}(v + z, u + z) \neq p_{ij}(v, u)$$

or

$$p^*_{ij}(v + z, u + z) \neq p^*_{ij}(v, u)$$

Such a non-stationary description of the transition probability corresponding to the defined random processes, $N(x)$ and $N^*(x)$, in effect, captures the mutual dependence between the peak amplitude levels. This will, therefore, be used to describe the statistical characteristics of all the peaks in the process, from which it

will be possible to obtain the statistics of the different ordered peaks. Now, to obtain the transition probability, let us first investigate the probability of the change in the state random variable corresponding to an infinitesimal change in the space random variable. These probabilities are, in fact, the infinitesimal generator of the process which generates the ordered peaks in the response process.

Let us consider for the first case an infinitesimal change h in the space parameter of the random process $N(x)$ from s to $s + h$. The probability rules based on Markov theory for an infinitesimal change to build up the random process $N(x)$ are as follows:

$$\operatorname{Prob}\{N(s+h) - N(s) = 1 \mid N(s) = i\} = p_{i,i+1}(s+h, s) = \lambda_{i,s}h + O_{1,i}(h, s) \tag{3}$$

$$\operatorname{Prob}\{N(s+h) - N(s) = 0 \mid N(s) = i\} = p_{ii}(s+h, s) = 1 - \lambda_{i,s}h + O_{2,i}(h, s) \tag{4}$$

$$\begin{aligned}\operatorname{Prob}\{N(s+h) - N(s) > 1 \mid N(s) = i\} &= p_{ij}(s+h, s)\\ &= O_{3,i}(h, s), \quad \forall j \geqslant i+2\end{aligned} \tag{5}$$

where $\lambda_{i,s} > 0$ is the infinitesimal generator of the process and

$$\lim_{h\to 0} \frac{O_{j,i}(h, s)}{h} = 0, \quad j = 1, 2, 3 \tag{6}$$

Now, the differential equation for the transition probability for the process can be derived as follows. We use the fact that the transition probability $p_{ij}(v, u)$ from level u to v can be decomposed into two stages $p_{ik}(s, u)$ and $p_{kj}(v, s)$ for all the possible values of state k at level s. Using the law of total probability, one can write

$$p_{ij}(v, u) = \sum_k p_{ik}(s, u)\, p_{kj}(v, s) \tag{7}$$

The process in our consideration is in fact a pure birth process since the state variables cannot decrease with the change in space variable. Hence, from equation (7), we can write

$$p_{ij}(v+h, u) = \sum_{k=i}^{j} p_{ik}(v, u)\, p_{kj}(v+h, v). \tag{8}$$

On using equation (5), equation (8) may be expanded as

$$p_{ij}(v+h, u) = p_{i,j-1}(v, u)\, p_{j-1,j}(v+h, v) + p_{ij}(v, u)\, p_{jj}(v+h, v) + O_{3,i}(h, v) \tag{9}$$

On substitution of equations (3) and (4), equation (9) becomes

$$\begin{aligned}p_{ij}(v+h, u) - p_{ij}(v, u) &= p_{i,j-1}(v, u)(\lambda_{j-1,v}h + O_{1,j-1}(h, v))\\ &\quad + p_{ij}(v, u)(-\lambda_{j,v}h + O_{2,j}(h, v)) + O_{3,i}(h, v)\end{aligned} \tag{10}$$

Dividing both sides of this equation by h and letting h tend to zero, the differential equation is obtained as

$$\frac{\partial}{\partial v} p_{ij}(v, u) = \lambda_{j-1,v}\, p_{i,j-1}(v, u) - \lambda_{j,v}\, p_{ij}(v, u) \tag{11}$$

By using the boundary condition $p_{ij}(u, u) = 0,\ j > i$, this can be solved to obtain the following recursive relationship:

$$p_{ij}(v, u) = \mathrm{e}^{-\int \lambda_{j,v}\,\mathrm{d}v} \int_u^v \lambda_{j-1,s}\, p_{i,j-1}(s, u)\, \mathrm{e}^{\int \lambda_{j,s}\,\mathrm{d}s}\, \mathrm{d}s \tag{12}$$

To proceed in a recursive manner, it is required to obtain the initial transition probability. For this, we must obtain the solution of the initial differential equation for which the state variable remains unchanged. Hence, with similar reasoning as for equation (9), we can write

$$p_{ii}(v+h, v) = p_{ii}(v, u)\, p_{ii}(v+h, v) \tag{13}$$

Using equations (3) and (4) and after similar manipulations as adopted for equation (11), we obtain the initial differential equation as

$$\frac{\partial}{\partial v} p_{ii}(v, u) = -\lambda_{i,v} p_{ii}(v, u) \tag{14}$$

The solution of this equation has the form

$$p_{ii}(v, u) = C e^{-\int \lambda_{i,r} \mathrm{d}r} \tag{15}$$

where C is a constant to be obtained from the boundary condition. It is definite that without any change in level u, the state variable would remain unchanged at i. Then, one can write the boundary condition as

$$p_{ii}(u, u) = 1 \tag{16}$$

The constant C is thus evaluated from equation (15) by putting $v = u$. On back substitution of C into equation (15), we thus get

$$p_{ii}(v, u) = e^{-\int_u^v \lambda_{i,s} \mathrm{d}s} \tag{17}$$

Using $p_{ii}(v, u)$ in equation (12) now, we can obtain the one-step transition probability as

$$p_{i,i+1}(v, u) = e^{-\int \lambda_{i+1,r} \mathrm{d}r} \int_u^v \lambda_{i,s} e^{-\int_u^s \lambda_{i,w} \mathrm{d}w} e^{\int \lambda_{i+1,s} \mathrm{d}s} \mathrm{d}s \tag{18}$$

Similarly, $p_{i,i+2}$ can be obtained using equation (18) in equation (12). Thus, in general, the expression for $p_{ij}(v, u)$ can be obtained as

$$p_{ij}(v, u) = e^{-\int \lambda_{j,r} \mathrm{d}r} \int_u^v g_1(s_1) \int_u^{s_1} g_2(s_2) \cdots \int_u^{s_{j-i-1}} g_{j-i}(s_{j-i}) \mathrm{d}s_{j-i} \cdots \mathrm{d}s_2 \mathrm{d}s_1 \tag{19}$$

with

$$g_k(s_k) = \lambda_{j-k,s_k} e^{-\int \lambda_{j-k,s_k} \mathrm{d}s_k} e^{\int \lambda_{j-k+1,s_k} \mathrm{d}s_k} \quad k = 1, 2, \ldots, (j - i - 1)$$

and

$$g_{j-i}(s_{j-i}) = \lambda_{i,s_{j-i}} e^{-\int_u^{s_{j-i}} \lambda_{i,w} \mathrm{d}w} e^{\int \lambda_{i+1,s_{j-i}} \mathrm{d}s_{j-i}}$$

For the second case of the transition probability, with the increase in the space variable from s to $s + h$, the probability that the state variable decreases from i to $i - 1$ is $\lambda^*_{i,s}h$ and the probability that it stays at i is $(1 - \lambda^*_{i,s}h)$ for a death process. Hence, by following similar treatment as in the earlier case, one can obtain the backward differential equation for the transition probability as

$$\frac{\partial}{\partial v} p^*_{ij}(v, u) = \lambda^*_{j+1,v} p^*_{i,j+1}(v, u) - \lambda^*_{j,v} p^*_{ij}(v, u) \tag{20}$$

The initial differential equation, the boundary condition and its solution are same as in the first case. Hence, the generalized solution for $p^*_{ij}(v, u)$ is also similarly obtained as

$$p^*_{ij}(v, u) = e^{-\int \lambda^*_{j,r} \mathrm{d}r} \int_u^v g^*_1(s_1) \int_u^{s_1} g^*_2(s_2) \cdots \int_u^{s_{i-j-1}} g^*_{i-j}(s_{i-j}) \mathrm{d}s_{i-j} \cdots \mathrm{d}s_2 \mathrm{d}s_1 \tag{21}$$

with

$$g^*_k(s_k) = \lambda^*_{j+k,s_k} e^{-\int \lambda^*_{j+k,s_k} \mathrm{d}s_k} e^{\int \lambda^*_{j+k-1,s_k} \mathrm{d}s_k}, \quad k = 1, 2, \ldots, (i - j - 1)$$

and

$$g^*_{i-j}(s_{i-j}) = \lambda^*_{i,s_{i-j}} e^{-\int_u^{s_{i-j}} \lambda^*_{i,w} \mathrm{d}w} e^{\int \lambda^*_{i-1,s_{i-j}} \mathrm{d}s_{i-j}}$$

In the above expressions for $p_{ij}(v, u)$ and $p^*_{ij}(v, u)$, there is a dependence on the parameter λ. This parameter depends both on the earlier state and space values of the random variable and thus captures the dependence of a peak to be formed on the number of peaks formed below or above that level as is the case for $p_{ij}(v, u)$ or $p^*_{ij}(v, u)$, respectively. On proper estimation of the parameter λ, the transition probabilities for the above two cases can be solved. The transition probabilities in turn lead to the statistics of the ordered peak amplitudes. It is shown below how the different expressions for λ lead to the order statistics of the dependent and independent peak amplitudes and to a form of the first passage problem in time (when the probability of no maxima formation above a level is considered).

3. DISTRIBUTIONS FOR DIFFERENT CASES OF λ

To have an estimation of λ in different cases, we must obtain the probabilities $p_{i,i+1}(v + h, v)$ and $p^*_{i,i-1}(v + h, v)$ with the density functions, respectively, as $\lambda_{i,v}$ and $\lambda^*_{i,v}$. Let us concentrate on the random process $X(t)$ which is assumed to be a zero-mean, stationary, Gaussian process with a total of n peaks occurring in a duration T. The probability density function of these peaks as normalized with respect to the root-mean-square (r.m.s.) value of $X(t)$, a_{rms}, is given as (see Reference 14)

$$p(\eta) = \frac{1}{\sqrt{2\pi}}\left[\varepsilon \mathrm{e}^{-\eta^2/2\varepsilon^2} + (1 - \varepsilon^2)^{1/2}\eta \mathrm{e}^{-\eta^2/2}\int_{-\infty}^{\eta(1-\varepsilon^2)^{1/2}/\varepsilon} \mathrm{e}^{-x^2/2}\,\mathrm{d}x\right] \tag{22}$$

with

$$\varepsilon = \left[\frac{m_0 m_4 - m_2^2}{m_0 m_4}\right]^{1/2} \tag{23}$$

Here, m_0, m_2 and m_4, respectively, denote the zeroth, second and the fourth moments of the energy spectrum $E(\omega)$ of $X(t)$. The cumulative probability that the height of a maximum will be less than η can be written as

$$P(\eta) = \int_{-\infty}^{\eta} p(u)\,\mathrm{d}u \tag{24}$$

3.1. Order statistics of dependent peaks

Since $p_{i,i+1}(v + h, v)$ is to be obtained based on the condition that i peaks among n have occurred below v, then we can say that $(n - i)$ peaks are yet to occur. Thus, assuming that all the remaining $(n - i)$ peaks are identically distributed and that $p(v)h$ represents the probability of the occurrence of a peak between v and $v + h$, we can write

$$p_{i,i+1}(v + h, v) = (n - i)p(v)h + O_{1,i}(h, v) \tag{25}$$

Since $p^*_{i,i-1}(v + h, v)$ is based on the condition that i peaks are yet to occur above v, we can similarly write

$$p^*_{i,i-1}(v + h, v) = ip(v)h + O_{1,i}(h, v) \tag{26}$$

The above two equations give the expressions for $\lambda_{i,v}$ and $\lambda^*_{i,v}$ as

$$\lambda_{i,v} = (n - i)p(v) \tag{27}$$

and

$$\lambda^*_{i,v} = ip(v) \tag{28}$$

Corresponding transition probabilities (see equations (19) and (21)) thus become

$$p_{ij}(v, u) = \frac{(n - i)!}{(j - i)!(n - j)!}\mathrm{e}^{-(n-j)P(v)}\mathrm{e}^{(n-i)P(u)}[\mathrm{e}^{-P(u)} - \mathrm{e}^{-P(v)}]^{j-i} \tag{29}$$

and

$$p^*_{ij}(v, u) = \frac{i!}{(i-j)!\,j!} e^{-jP(v)} e^{iP(u)} [e^{-P(u)} - e^{-P(v)}]^{(i-j)} \tag{30}$$

Since, we are interested in obtaining the probability that the ith-order peak is below a level x, let us consider the case where $(n-j+1)$ peaks are below x and $(j-1)$ peaks are above x. This probability thus follows from the two transition probabilities obtained from equations (29) and (30) as

$$p_{0,n-j+1}(x, -\infty) = \frac{n!}{(j-1)!\,(n-j+1)!} e^{-(j-1)P(x)} [1 - e^{-P(x)}]^{n-j+1} \tag{31}$$

and

$$p^*_{j-1,0}(\infty, x) = e^{(j-1)P(x)} [e^{-P(x)} - e^{-1}]^{j-1} \tag{32}$$

It may be recalled that there are a total number of n peaks in the process, and all of them lie in the range $(-\infty, \infty)$. Thus, the probability distribution of the ith-order peak, $P_{X_{(i)}}(x)$, is equivalent to the probability of occurrence of the event in which no peak lies above ∞, ith-order peak lies below x and no peak lies below $-\infty$ on the condition that n peaks are present in the range $(-\infty, \infty)$. Using the fact that the ith-order peak below x implies that at least $(n-i+1)$ peaks are below x or at most $(i-1)$ peaks are above x, $P_{X_{(i)}}(x)$ may be expressed in the following form:

Prob(no peak above ∞, ith peak below x and no peak below $-\infty | n$ peaks are between $-\infty$ to ∞)
$= \sum_{j=1}^{i}$ Prob(no peak above $\infty \,|\, (j-1)$ or less peaks above x, n peaks above $-\infty$)
$\times$ Prob ($(n-j+1)$ or more peaks below $x \,|$ no peak below $-\infty$)/Prob (n peaks below $\infty \,|$ no peak below $-\infty$).

This condition may be expressed mathematically as

$$P_{X_{(i)}}(x) = \frac{\sum_{j=1}^{i} p^*_{j-1,0}(\infty, x)\, p_{0,n-j+1}(x, -\infty)}{p_{0,n}(\infty, -\infty)} \tag{33}$$

On using equations (31) and (32), this expression simplifies to

$$P_{X_{(i)}}(x) = \sum_{j=1}^{i} \frac{n!}{(j-1)!\,(n-j+1)!} \left[\frac{e^{-P(x)} - e^{-1}}{1 - e^{-1}}\right]^{j-1} \left[\frac{1 - e^{-P(x)}}{1 - e^{-1}}\right]^{n-j+1} \tag{34}$$

On differentiation, this yields the density function for the ith-order peak as

$$p_{X_{(i)}}(x) = \frac{n!}{(n-i)!\,(i-1)!} \left[\frac{e^{-P(x)} - e^{-1}}{1 - e^{-1}}\right]^{i-1} \left[\frac{1 - e^{-P(x)}}{1 - e^{-1}}\right]^{n-i} \frac{e^{-P(x)}}{1 - e^{-1}} p(x) \tag{35}$$

3.2. *Order statistics for independent peaks*

When the peaks are assumed to be statistically independent, $\lambda_{i,v}$ or $\lambda^*_{i,v}$ for the two cases of transition probabilities can be obtained from $p_{i,i+1}(v+h, v)$ or $p^*_{i,i-1}(v+h, v)$, simply by using the probability that one maximum is formed in the range v to $v+h$ irrespective of the peaks which have already occurred or are yet to occur. Thus, we can write

$$p_{i,i+1}(v+h, v) = np(v)h + O_{1,i}(h, v) \tag{36}$$

and

$$p^*_{i,i-1}(v+h, v) = np(v)h + O_{1,i}(h, v) \tag{37}$$

which for both the cases of transition probabilities lead to

$$\lambda_{i,v} \quad \text{or} \quad \lambda^*_{i,v} = np(v) \tag{38}$$

Thus, on using equations (19) and (21) and integrating, we obtain the transition probabilities as

$$p_{ij}(v, u) = n^{j-i} e^{-[P(v)-P(u)]} \frac{[P(v) - P(u)]^{j-i}}{(j-i)!} \tag{39}$$

and

$$p^*_{ij}(v, u) = n^{i-j} e^{-[P(v)-P(u)]} \frac{[P(v) - P(u)]^{i-j}}{(i-j)!} \tag{40}$$

For the ith-order peak, we obtain the two transition probabilities (as was done for the dependent peaks, in Section 3.1) as

$$p_{0,n-j+1}(x, -\infty) = n^{n-j+1} e^{-P(x)} \frac{[P(x)]^{(n-j+1)}}{(n-j+1)!} \tag{41}$$

and

$$p^*_{j-1,0}(\infty, x) = n^{j-1} e^{[P(x)-1]} \frac{[1 - P(x)]^{(j-1)}}{(j-1)!} \tag{42}$$

On proceeding further, we obtain the distribution and the density functions, $P_{X_{(i)}}(x)$ and $p_{X_{(i)}}(x)$, respectively as

$$P_{X_{(i)}} = \sum_{j=1}^{i} \frac{n!}{j!(n-j)!} P(x)^{(n-j+1)} [1 - P(x)]^{j-1} \tag{43}$$

and

$$p_{X_{(i)}} = \frac{n!}{(n-i)!(i-1)!} P(x)^{(n-i)} [1 - P(x)]^{(i-1)} p(x) \tag{44}$$

The above expressions in equations (43) and (44) exactly match with the known distribution and the density functions for the order statistics of the independent peaks (e.g., see References 2 and 15).

3.3. *Probability of no peak above a specified level in a given time*

Contrary to the earlier two cases, here we do not directly deal with the total number of the peaks. This case rather involves time as a variable. Since the product of the time rate of occurrences of peaks with the total time gives the total number of peaks, it will be convenient to consider the time rate of occurrences of maxima in the range v to $v + h$ to introduce the time as a variable in the problem. Under the assumption of stationarity along the time axis and the independence of peak occurrences, we can write

$$p_{i,i+1}(v + h, v) = N_1 p(v) Th + O_{1,i}(h, v) \tag{45}$$

where N_1 is the frequency of occurrence of maxima and is expressed in terms of the moments of the energy spectrum $E(\omega)$ as (see Reference 14)

$$N_1 = \frac{1}{2\pi} \left[\frac{m_4}{m_2} \right]^{1/2} \tag{46}$$

From equation (45) we obtain,

$$\lambda_{i,v} = N_1 p(v) T \tag{47}$$

which, on substitution into equation (19) and on integration, yields

$$p_{ij}(v, u) = (N_1 T)^{j-i} e^{-N_1[P(v) - P(u)]T} \frac{[P(v) - P(u)]^{j-i}}{(j-i)!} \tag{48}$$

Now, to obtain the probability that no maximum is formed above the level x in time T, we evaluate the probability that i peaks will be formed below ∞ on the condition that the same number of peaks had occurred below x. Thus, the probability of no formation of maximum above the amplitude level x is equivalently given by

$$p_{ii}(\infty, x) = e^{-N_1[1 - P(x)]T} \tag{49}$$

This expression is of the same form as the general expression of the first passage probability, written as

$$P(T) = Ae^{-\alpha T} \tag{50}$$

Here, A is obtained from the condition whether the process, initially at time $t = 0$, is above or below the specified level (see Reference 13). Based on that, equation (49) can also be modified. However, the rate parameter α denoting the rate of crossings in the case of first passage problem differs from the rate in equation (49). In fact, the rate parameter as in equation (49) is associated with the formation of maxima, not with the rate of crossings.

There is a subtle difference between the first passage problem and the problem of 'no occurrence of peaks' as considered above in this subsection. In the first passage problem, we are concerned with time as the parameter and the probability corresponds to the event of first crossing of a level at the time instant $t = T$. On the other hand, here we have the probability of 'no peaks' above a specified level in the time duration T and for this case the critical level may or may not be crossed at time $t = T$. However, safe operation of a structure does not necessarily imply the crossing of the critical level at time $t = T$. If the final response is bounded, the critical collapse level will not at all be crossed by the response process. Then, the case of 'no peaks' will provide a more realistic assessment of the structural safety as compared to the first passage problem. The first passage problem may be more relevant to those applications, e.g. the structural response to extremely severe earthquakes, where the failure of the system is deemed to occur as soon as the system response crosses the critical level. Since the civil engineering structures are so designed that those behave in a ductile manner and their responses are associated with the crossings of the linear design level taking place several times, not just once, the order statistics formulation proposed in this paper becomes more meaningful compared to the first passage problem.

4. RESULTS AND DISCUSSION

A digital simulation of the strong ground motion as a random process has been carried out by generation of synthetic accelerograms for the Dumbarton Bridge site (near Coyote Hills) and corresponding to the 1989 Loma Prieta Earthquake (see Reference 16 for details). 140 records with the total duration T of 40·96 sec have been obtained and stationary segments of these records have been considered by curtailing the tails with $t < 8$ sec and with $t > 20$ sec (as in Reference 1). The ε values of these records have been computed from the ratio of number of negative to total number of peaks (see Reference 14), and the records have been grouped in three groups with the average ε values equal to 0·5, 0·6 and 0·7, respectively. There were 40, 63 and 83 records, respectively, in these groups. The total number of peaks was, on average, 80 for all the three groups. For each group, amplitudes of the first-, fifth- and ninth-order peaks have been normalized with respect to the r.m.s. values of the respective records, and histograms have been obtained for these normalized amplitudes. As shown in Figures 1–9 by dotted lines, these have been compared with the results obtained from the analytical formulation.

The density functions obtained from the synthetic data are seen to match reasonably with the theoretical curves, except for an apparent shift along the random variable axis. To see whether this has resulted due to

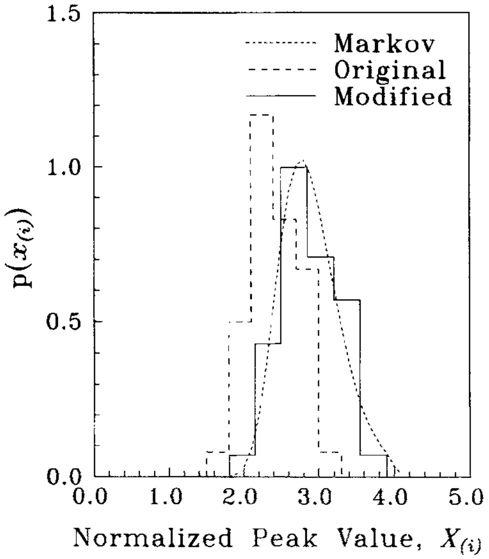

Figure 1. Comparison of Markov and experimental p.d.f. for $i = 1$, $\varepsilon = 0{\cdot}5$, $n = 80$

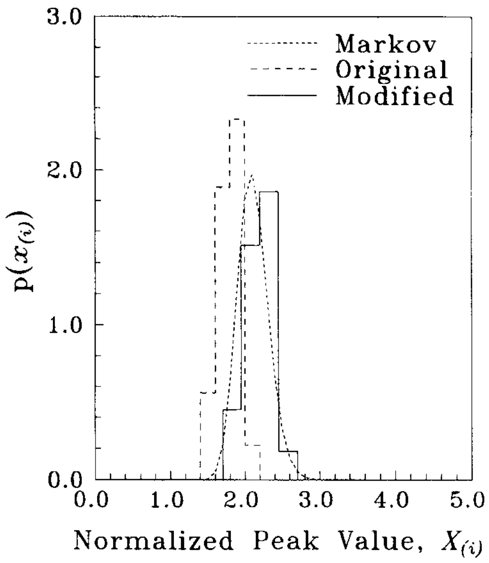

Figure 2. Comparison of Markov and experimental p.d.f. for $i = 5$, $\varepsilon = 0{\cdot}5$, $n = 80$

the error in normalization of the peak amplitudes, ratios of expected amplitudes of the fifth- and ninth-order peaks to the expected amplitude of the first-order peak (i.e. $E(X_{(5)})/E(X_{(1)})$ and $E(X_{(9)})/E(X_{(1)})$) have been compared for the theoretical and experimental results in Table I. Table II shows the comparison of the standard deviations for the first-, fifth- and ninth-order peaks. It is seen from Table I that there is excellent agreement between the theoretical and experimental results whereas in Table II, these results are quite close. To illustrate this further, histograms in Figures 1–9 have been plotted again (in solid lines) by modifying the r.m.s. values of the records such that the means of the histograms match with the means obtained from the formulation in the case of the first-order peaks. The excellent agreement of the new histograms with the

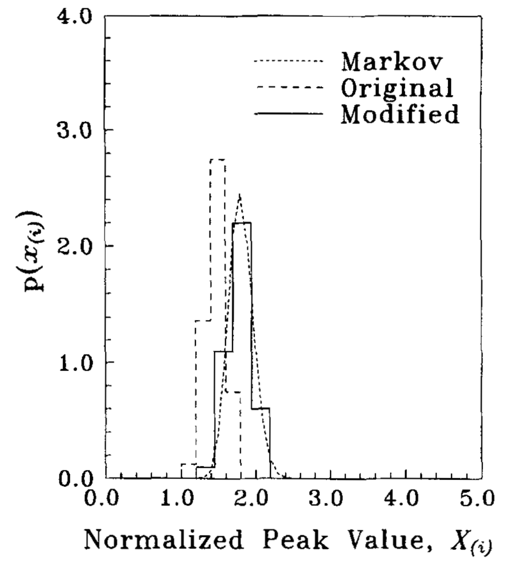

Figure 3. Comparison of Markov and experimental p.d.f. for $i = 9$, $\varepsilon = 0{\cdot}5$, $n = 80$

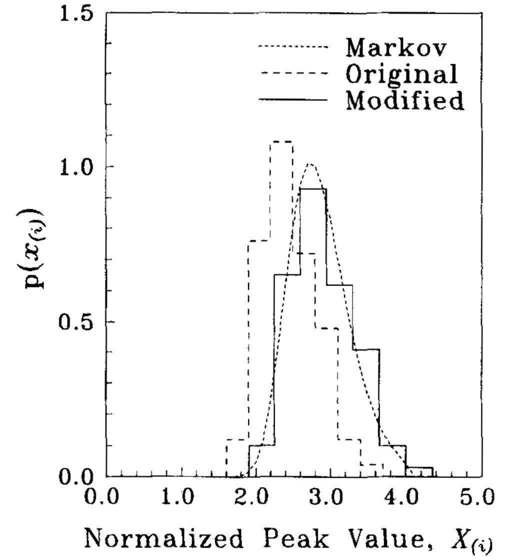

Figure 4. Comparison of Markov and experimental p.d.f. for $i = 1$, $\varepsilon = 0{\cdot}6$, $n = 80$

density curves confirms that the r.m.s. value of the process might have had some effect of residual non-stationarity in the records, leading to the error in normalization of peaks.

The analytical expression for the probability density functions of the ordered peaks (see equation (35)) is now used to obtain the density functions for the different orders of peaks. For the parametric variation, different values of the total number of peaks n and different values of the parameter ε representing the bandwidth of the frequency content of the process have been considered. Figure 10 shows the probability density curves corresponding to the first-, fifth- and ninth-order peaks for a process with $\varepsilon = 0{\cdot}4$ and $n = 160$. These values are representative of a typical earthquake response process. The curves show that the expected

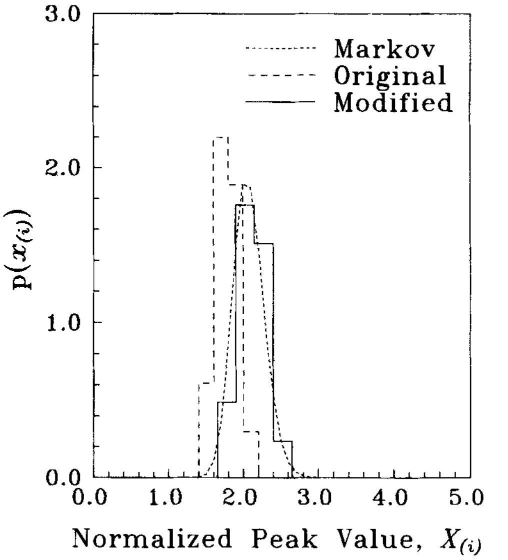

Figure 5. Comparison of Markov and experimental p.d.f. for $i = 5$, $\varepsilon = 0{\cdot}6$, $n = 80$

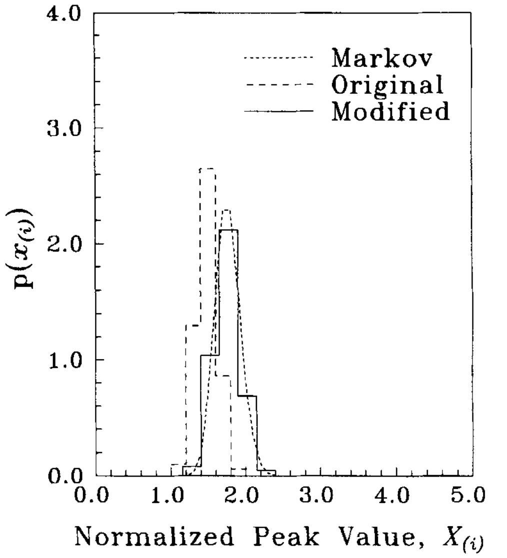

Figure 6. Comparison of Markov and experimental p.d.f. for $i = 9$, $\varepsilon = 0{\cdot}6$, $n = 80$

value decreases and the density function becomes more sharply peaked with the increase in the order of peak. This suggests that the expected value can be used as a reasonable measure for estimating the higher-order peak amplitudes while for the lower orders the variance may also have to be specified for describing adequately the statistical characteristics of the first few orders of peaks. To see the effect of the bandwidth of the process, probability density curves have been obtained (see Figure 11) for the first-order ($i = 1$) peak, corresponding to $\varepsilon = 0{\cdot}0, 0{\cdot}4, 0{\cdot}8, 0{\cdot}9, 0{\cdot}95$ and $1{\cdot}0$. It may be noted that $\varepsilon = 0{\cdot}0$ and $1{\cdot}0$ represent two extreme cases of narrow-band and wide-band processes (see Reference 17). The response of the lightly damped structures to earthquake excitations is one example of the narrow-band processes. It is seen from the figure

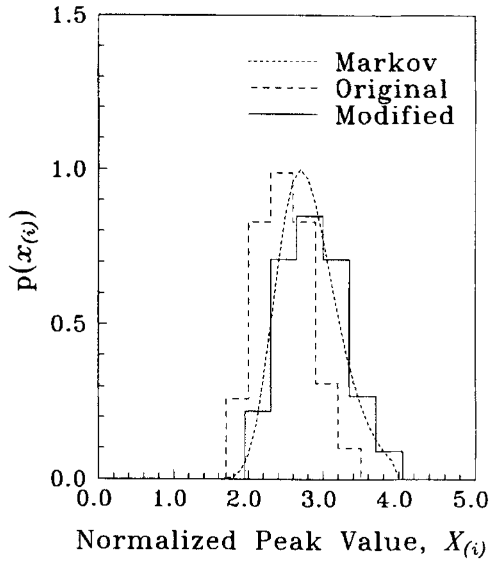

Figure 7. Comparison of Markov and experimental p.d.f. for $i = 1$, $\varepsilon = 0{\cdot}7$, $n = 80$

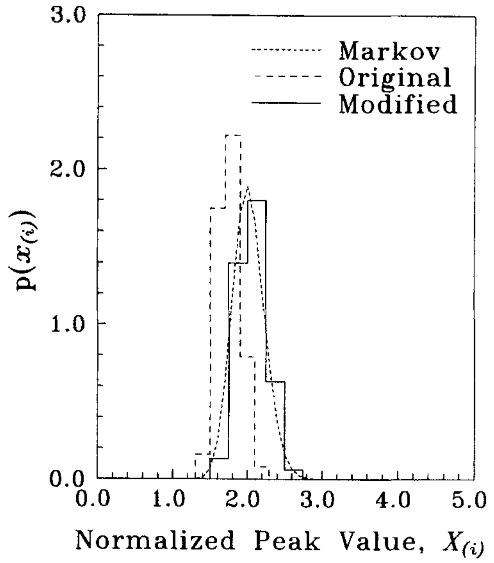

Figure 8. Comparison of Markov and experimental p.d.f. for $i = 5$, $\varepsilon = 0{\cdot}7$, $n = 80$

that the dispersion is almost invariant except for a marginal increase with ε beyond 0·4. Thus, for most structural response processes, this variation may be neglected. However, it may be noted that irrespective of the order of peak, the expected value decreases substantially with the increase in the value of ε beyond 0·4 while it remains almost the same for narrow-band processes with $\varepsilon < 0{\cdot}4$. To see this more clearly for the other orders of peaks, variation of normalized expected amplitudes has been plotted with the order of peak (see Figure 12). Three curves have been obtained corresponding to $\varepsilon = 0{\cdot}0$, 0·4, and 1·0. These curves are also compared and found to be in broad agreement with the curves obtained by Gupta and Trifunac[2] for statistically independent peaks. Here it becomes obvious that for any order of peak, assuming the structural

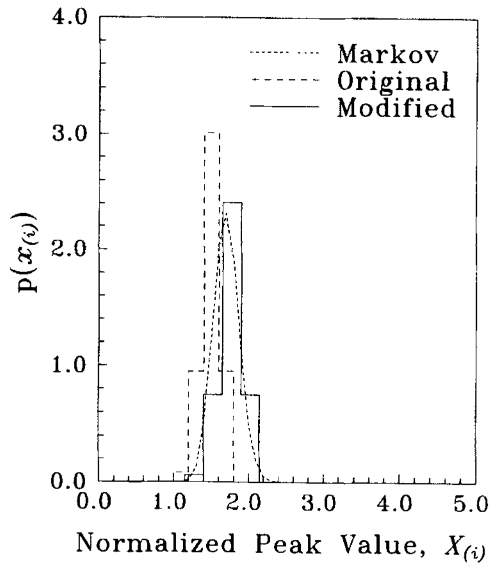

Figure 9. Comparison of Markov and experimental p.d.f. for $i = 9$, $\varepsilon = 0{\cdot}7$, $n = 80$

Table I. Comparison of $E(X_{(i)})/E(X_{(1)})$ from experiment and theory

	Fifth order peak ($i = 5$)		Ninth order peak ($i = 9$)	
ε	Experiment	Theory	Experiment	Theory
0·5	0·74	0·73	0·62	0·62
0·6	0·72	0·74	0·61	0·63
0·7	0·71	0·73	0·60	0·61

Table II. Comparison of standard deviations from experiment and theory

	First order peak		Fifth order peak		Ninth order peak	
ε	Experiment	Theory	Experiment	Theory	Experiment	Theory
0·5	0·34	0·42	0·15	0·20	0·16	0·16
0·6	0·37	0·42	0·16	0·21	0·15	0·17
0·7	0·36	0·43	0·16	0·21	0·13	0·17

response processes (with $\varepsilon < 0{\cdot}4$) to be narrow-banded is an excellent approximation. The same thing cannot, however, be done for the input excitation processes (with ε between 0·6 and 0·8), particularly in the case of the higher-order peaks. These trends are also observed for the results based on the alternative formulation (using simulated joint density) as shown in Reference 1. It may further be observed in Figure 12 that the normalized peak amplitudes for statistically independent peaks deviate more from the Markovian estimates with the increase in ε, particularly for the higher orders of peaks. Those agree better for the first few orders of peaks. In fact, it has been shown in Reference 1 through several examples that it may be acceptable to ignore the peak dependence in estimating peak factors for these peaks.

The effect of number of peaks on the probability densities of the ordered response peaks has been studied by plotting the probability density curves for $n = 40, 80, 120, 160$ with ε taken as 0·4. Figure 13 shows these

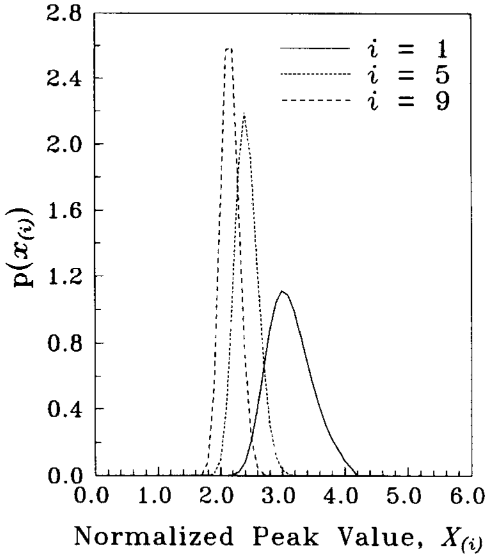

Figure 10. Comparison of p.d.f. of different ordered peaks for $\varepsilon = 0{\cdot}4$ and $n = 160$

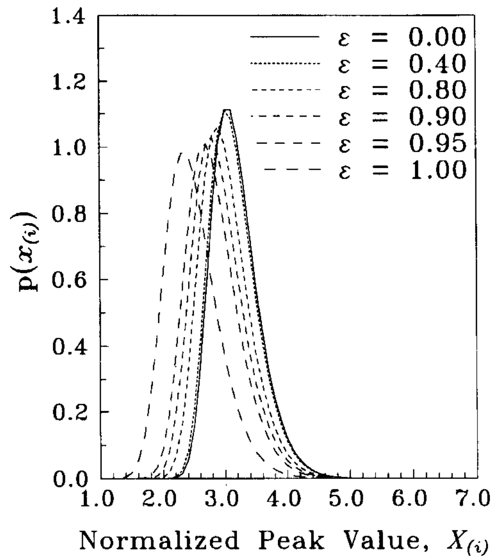

Figure 11. Comparison of p.d.f. for first-order peak with $n = 160$

results for the first-order ($i = 1$) peak. It may be noted that the dispersion in the curves is higher for the cases of smaller numbers of total peaks. This sounds logical since with the smaller number of peaks, the different peak amplitudes will be farther spaced on an 'average' basis, and thus there will be a greater range for their possible variations. Further, it is seen that the expected value of an ordered peak increases with the increase in the total number of peaks. Since the total number of peaks in any response process of a particular structural system primarily depends on the duration of the excitation, this means that larger (normalized) amplitudes can be expected for the ordered peaks in the case of longer duration earthquakes. This increase in the expected value of the ordered peak does not, however, seem to be uniform with the increase in the number

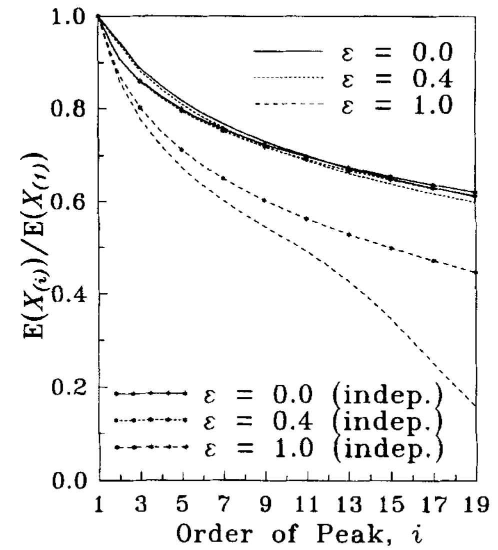

Figure 12. Variation in normalized peak amplitudes with the order of peak for $n = 160$

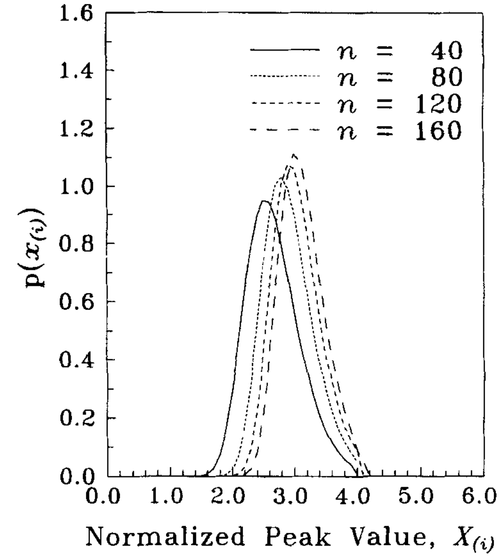

Figure 13. Density functions for largest order of peak with $\varepsilon = 0{\cdot}4$

of peaks. To ascertain this, the expected amplitudes of the first-, fifth- and ninth-order peaks have been plotted for different values of number of peaks n (see Figure 14). It is seen that for greater values of n, say between 120 and 200, expected amplitudes of any particular order of peak vary little with the variation in n. For this range of n, the approach of taking constant (largest) peak factor equal to 3 appears to be reasonably good. For the short-duration earthquakes and for the long-period structures, this should, however, give slightly conservative estimates. A similar trend has been observed with the estimates from the formulations based on simulated joint density and independence of peaks (see Reference 1). Further, it can be seen that in case of the large number of peaks, greater structural damage associated with the longer-duration excitations

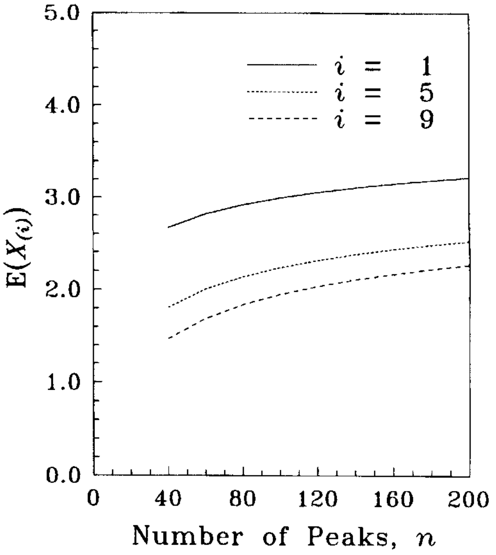

Figure 14. Variation in ordered expected value with n for $\varepsilon = 0{\cdot}4$

takes place largely due to the excursions by the lower-order peaks, not due to the marginally increased amplitude of the largest peak.

5. CONCLUSIONS

The probability distributions for the ordered peak amplitudes in a random process have been obtained using the transition probabilities in the Markov theory. The theoretical results are seen to match reasonably well with the digital experimentation results based on 140 synthetic accelerograms. The analytical expressions are simple and closed form in nature, and thus can easily be handled without much computational difficulty. Apart from being able to account for the dependence between the peak amplitudes in a generalized form, the proposed approach integrates the two apparently uncorrelated approaches, one based on the first passage time and the other on the order statistics of the amplitude level. Though this approach is formulated for the stationary processes, it can also be applied to the transient response processes by the estimation of a 'working' r.m.s. value through response spectrum approach (e.g. see Reference 18).

It has been seen from the parametric study that the density functions and the expected values of higher-order peaks do not vary much in the practical range of ε for the response of structures. Thus, a narrow-band approximation for computing the peak factors in case of most earthquake responses seems to be acceptable. Further, the higher-order peak amplitudes are represented accurately by their expected values for any given level of confidence.

The formulation in this paper is based on the concept of transition probabilities. It may be possible to extend this concept further to obtain more statistical information about the process which may then be used in the stochastic studies on earthquake engineering, particularly those on the structural damage analysis.

REFERENCES

1. B. Basu, V. K. Gupta and D. Kundu, 'Ordered peak statistics through digital simulation', *Earthquake eng. struct. dyn.* **25**, 1061–1073 (1996).
2. I. D. Gupta and M. D. Trifunac, 'Order statistics of peaks in earthquake response', *J. eng. mech. div. ASCE* **114**(10), 1605–1627 (1988).
3. S. H. Crandall, K. L. Chandiramani and R. G. Cook, 'Some first passage problems in random vibration', *J. appl. mech. ASME* **88**(E), 532 (1966).

4. S. H. Crandall, 'First-crossing probabilities of the linear oscillator', *J. sound vib.* **12**(3), 285 (1970).
5. W. D. Mark, 'On false-alarm probabilities of filtered noise', *Proc. IEEE* **54**, 316 (1966).
6. R. G. Cook, 'Digital simulation of random vibrations', *Doctoral Thesis*, Department of Mechanical Engineering, M.I.T., Cambridge, Massachusetts, 1964.
7. O. Ditlevesen, 'Extremes and first passage time with applications in civil engineering', *Doctoral Thesis*, Technical University of Denmark, Copenhagen, Denmark, 1971.
8. H. Cramer, 'On the intersections between the trajectories of a normal stationary stochastic process and a high level', *Arkiv. mat.* **6**, 337 (1966).
9. Y. K. Lin, 'First-excursion failure of a randomly excited structure', *AIAA j.* **8**(4), 720 (1970).
10. J. N. Yang and M. Shinozuka, 'On the first-excursion probability in stationary narrow-band random vibration', *J. appl. mech. ASME* **38**(4), 1017 (1971).
11. D. A. Darling and A. J. F. Siegert, 'The first passage problem for a continuous process', *Ann. math. statist.* **24**, 624 (1953).
12. E. H. Vanmarcke, 'Properties of spectral moments with applications to random vibration', *J. eng. mech. div., Proc. ASCE* **98** (EM2), 425–446 (1972).
13. E. H. Vanmarcke, 'On the distribution of the first-passage time for normal stationary random processes', *J. appl. mech. ASME* **42**(E), 215–220 (1975).
14. D. E. Cartwright and M. S. Longuet-Higgins, 'The statistical distribution of maxima of a random function', *Proc. roy. soc. London* **A 237**, 212–232 (1956).
15. H. A. David, *Order Statistics*, Wiley, New York, 1980.
16. V. K. Gupta and M. D. Trifunac, 'A note on the effects of ground rocking on the response of buildings during 1989 Loma Prieta earthquake', *Earthquake eng. eng. vib.* **13**(2), 12–28 (1993).
17. F. E. Udwadia and M. D. Trifunac, 'Characterization of response spectra through the statistics of oscillator response', *Bull. seism. soc. Amer.* **64**, 205–219 (1974).
18. V. K. Gupta and M. D. Trifunac, 'Response of multistoried buildings to ground translation and rocking during earthquakes', *J. prob. eng. mech.* **5**(3), 138–145 (1990).